Complex Geometry Exercises

Week 6

Exercise 1. *For all $n > 0, k \in \mathbb{Z}$, compute*

$$H^n\big(\mathbb{CP}^n, \mathcal{O}(k)\big) \ .$$

Exercise 2. *Let $\big(V^{2n}, \langle \cdot\,, \cdot \rangle\big)$ be a Euclidean vector space.*

(i) Show that the space of compatible complex structure is parametrised by two disjoint copies of $\mathrm{SO}(2n)/\,\mathrm{U}(n)$.

(ii) Show that for $n = 2$, this corresponds to two copies of $\mathbb{CP}^1$.

(iii) Show that for $n = 3$, this corresponds to two copies of $\mathbb{CP}^3$.

*(**Hint:** Recall the exceptional isomorphisms* $\mathrm{Spin}(4) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)$ *and* $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$*.)*

For the remainder of the sheet, let $\big(V^{2n}, \langle \cdot\,, \cdot \rangle, I\big)$ denote a Euclidean vector space of real dimension $2n$ with compatible complex structure I.

Exercise 3. *Let $p, q, p', q' \in \mathbb{N}$ with $p + q = k = p' + q'$ and $k \leq n$.*
Consider the Hodge-Riemann pairing:

$$\begin{aligned} Q : {\bigwedge}^{p,q} V_{\mathbb{C}}^* \times {\bigwedge}^{p',q'} V_{\mathbb{C}}^* &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto (-1)^{\binom{k}{2}} \alpha \wedge \beta \wedge \omega^{n-k} \ . \end{aligned}$$

Show

(i) Q vanishes unless $(p, q) = (q', p')$.

(ii) For $0 \neq \alpha \in P^{p,q} \subseteq \Lambda^{p,q} V_{\mathbb{C}}^*$, we have

$$i^{p-q} Q(\alpha, \overline{\alpha}) = [n - (p+q)]!\ \langle \alpha, \alpha \rangle > 0 \ .$$

(continues on the back)

Exercise 4. *Let $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ be an orthonormal basis of V. Show that, for any $\alpha \in \Lambda^k V$*

$$\Lambda\alpha(X_1, \ldots, X_{k-2}) = \sum_{i=1}^{n} \alpha(x_i, y_i, X_1, \ldots, X_{k-2}) \ .$$

Exercise 5. *Let $(E, \overline{\partial}_E, h)$ be a holomorphic hermitian vector bundle on X. Show that*

(i) the space of connections on E is an affine space modelled on $\mathcal{A}^1_X\big(\operatorname{End}(E)\big)$;

(ii) the space of metric connections on E is an affine space modelled on $\mathcal{A}^1_X\big(\operatorname{End}(E,h)\big)$;

(iii) the space of compatible connections on E is an affine space modelled on $\mathcal{A}^{1,0}_X\big(\operatorname{End}(E)\big)$.

Use the above to give an alternate proof of the uniqueness of the Chern connection.

Exercise 6. *Prove that*

(i) $\overline{\partial}^* = - * \partial *$ and $\partial^* = - * \overline{\partial} *$,

(ii) $\Delta_{\overline{\partial}} = \overline{\Delta_{\partial}}$.

(iii) $\mathcal{H}^{p,q}_{\overline{\partial}} = \ker \overline{\partial} \ \cap \ker \overline{\partial}^*$.

Exercise 7. *The goal of this exercise is to establish a Hodge isomorphism for vector bundles. Let $(E, \overline{\partial}_E, h)$ be a holomorphic hermitian vector bundle.*

(i) Show that the Hodge star extends naturally to E-valued forms.

(ii) Show that $\overline{\partial}^*_E = - \ \overline{*}\overline{\partial}_E\overline{*}$ is an L^2-adjoint to $\overline{\partial}_E$ with respect to the L^2-inner product induced by h.

(iii) Assuming we have an analogous Hodge decomposition, reproduce the argument in the lectures to conclude there is an isomorphism

$$H^q(X, \Omega^p \otimes E) \cong \mathcal{H}^{p,q}(X, E) \ ,$$

where $\mathcal{H}^{p,q}(X,E) \coloneqq \ker \Delta_{\overline{\partial}_E}$.